Derivation of pure-birth model

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Consider a lineage that exists (e.g., the opposite of *not existing*) with probability $P(t)$. We say that the lineage does not exist (in its current state, anyway) if it undergoes a speciation event. Allow that the lineage can undergo speciation on some time interval Δt with some probability $\lambda \Delta t$.

It must then be true that, on the same time interval Δt, the lineage will *not undergo speciation* with probability $1 - \lambda \Delta t$.

We can then write down an equation for the *new probability* that the lineage exists in its current state some amount of time Δt later, as

$$P(t + \Delta t) = (1 - \Delta t \lambda) P(t) \tag{1}$$

So, the probability of the lineage in its current state some time Δt later is simply the current probability of existence $P(t)$ multiplied by the probability that nothing happens on the focal interval. We can rearrange this equation to:

$$P(t + \Delta t) = P(t) - \Delta t \lambda P(t) \tag{2}$$

and then

$$P(t + \Delta t) - P(t) = -\Delta t \lambda P(t) \tag{3}$$

Now, we can maybe see where this is going. Remember from calculus that the definition of a derivative is:

$$\frac{dX}{dt} = \lim_{\Delta t \to 0} \frac{X(t + \Delta t) - X(t)}{\Delta t} \tag{4}$$

We want to make a differential equation for the change in probability as a function of time, dP/dt, and we can do this by dividing both sides by Δt and taking limits as $\Delta t \to 0$:

$$\frac{P(t + \Delta t) - P(t)}{\Delta t} = \frac{-\Delta t \lambda P(t)}{\Delta t} \tag{5}$$

$$\lim_{\Delta t \to 0} \frac{P(t+\Delta t) - P(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{-\Delta t \lambda P(t)}{\Delta t} \tag{6}$$

So this gives us

$$\frac{dP}{dt} = -\lambda P(t) \tag{7}$$

This is a simple differential equation that can be solved to yield an equation for the probability that a lineage *will not speciate* after some time t.

Representing $P(t)$ as P, we can make some simple rearrangements:

$$\frac{dP}{P} = -\lambda dt \tag{8}$$

We can solve this by integrating both sides:

$$\int \frac{dP}{P} = \int -\lambda dt \tag{9}$$

$$\ln(P) = -\lambda t + c \tag{10}$$

where c is the constant of integration. Exponentiating both sides:

$$P = e^{-\lambda t + c} = e^{-\lambda t} e^{c} \tag{11}$$

Rewriting with $P(t)$:

$$P(t) = e^{-\lambda t} e^{c} \tag{12}$$

To deal with the constant of integration, we note that we have the initial condition $P(0) = 1$, or

$$P(t) = 1 = e^{0} e^{c} \tag{13}$$

or

$$1 = e^{c} \tag{14}$$

So $c = 0$, and the probability of a given waiting time t_i, under a pure-birth process, is:

$$P(t_i) = e^{-\lambda t_i} \tag{15}$$